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UNIQUENESS OF MEROMORPHIC FUNCTIONS THAT SHARE ONE SMALL FUNCTION WITH THEIR DERIVATIVES

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Abstract

In this paper, we study the uniqueness theorem of meromorphic functions that share one small function with their derivatives, and obtain two theorems, which answer the question posed by Zhang and L*ü*, and improve and generalize the related results of Yu, Zhang et al..

1. Introduction and Main Results

In this paper, we use the standard notations and terms in the value distribution theory (see [5]). For convenience, we give the following notations and definitions. For any constant a, we denote by $N_{k)}(r, \frac{1}{f-a})$ the counting function for zeros of $f(z)-a$ with multiplicity no more than k, and by $\overline{N}_{k)}(r, \frac{1}{f-a})$, the corresponding

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one for which multiplicity is not counted. Let $N_{(k}(r, \frac{1}{f-a})$ be the counting function for zeros of $f(z)-a$ with multiplicity at least k and $\overline{N}_{(k}(r, \frac{1}{f-a})$ be the corresponding one for which multiplicity is not counted. Set $N_k(r, \frac{1}{f-a}) = \overline{N}(r, \frac{1}{f-a}) + \overline{N}_{(2}(r, \frac{1}{f-a}) + \cdots + \overline{N}_{(k}(r, \frac{1}{f-a})$. We define

$$\delta_k(a, f) = 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}_k(r, \frac{1}{f-a})}{T(r, f)}.$$

Obviously, $1 \geq \Theta(a, f) \geq \delta_k(a, f) \geq \delta(a, f) \geq 0$.

In additional, we shall also use the following notations:

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions such that $f(z)$ and $g(z)$ share 1 IM. We denote by $\overline{N}_L(r, \frac{1}{f-1})$ the counting function for 1-point of both $f(z)$ and $g(z)$ about which $f(z)$ has larger multiplicity than $g(z)$, with multiplicity being not counted, and denote by $N_{11}(r, \frac{1}{f-1})$ the counting function for common simple 1-point of both $f(z)$ and $g(z)$, and denote by $N_{(22}(r, \frac{1}{f-1})$ the counting function of those same multiplicity 1-point of both $f(z)$ and $g(z)$ and the multiplicity is ≥ 2. In the same way, we can define $\overline{N}_L(r, \frac{1}{g-1})$, $N_{11}(r, \frac{1}{g-1})$, and $N_{(22}(r, \frac{1}{g-1})$. Especially, if $f(z)$ and $g(z)$ share 1 CM, then $\overline{N}_L(r, \frac{1}{g-1}) = 0$.

Bruck considered the uniqueness problem of an entire function sharing one value with its derivative and proved the following theorem:

Theorem A ([1]). *Let* $f(z)$ *be a non-constant entire function. If f and* f' *share the value* 1 *CM and if* $N(r, \frac{1}{f'}) = S(r, f)$. *Then* $\frac{f'-1}{f-1} \equiv c$ *for some constant* $c \in C \setminus \{0\}$.

Yang [6] and Zhang [8] extended Theorem A and obtained many excellent theorems.

Theorem B ([6]). *Let* $f(z)$ *be a non-constant meromorphic function and k be a positive integer. Suppose that f and* $f^{(k)}$ *share* 1 *CM and*

$$2\overline{N}(r, f) + \overline{N}(r, \frac{1}{f'}) + N(r, \frac{1}{f^{(k)}}) < (\lambda + o(1))T(r, f^{(k)}),$$

for $r \in I$, *where* I *is a set of infinite linear measure and* λ *satisfies* $0 < \lambda < 1$, *then* $\frac{f^{(k)}-1}{f-1} \equiv c$ *for some constant* $c \in C \setminus \{0\}$.

Yu considered the problem of a meromorphic functions sharing one small function with its derivative and proved the following theorem:

Theorem C ([7]). *Let f be a non-constant meromorphic function and* $a(z)(\not\equiv 0, \infty)$ *be a small function with respect to f. If*

(1) *f and a have no common poles*,

(2) $f - a$ *and* $f^{(k)} - a$ *share the value* 0 *CM,*

(3) $4\delta(0, f) + 2(8 + k)\Theta(\infty, f) > 19 + 2k$,

then $f \equiv f^{(k)}$, *where k is a positive integer.*

In the same paper, Yu posed four open questions. Lahiri [2], Liu and Gu [3], and Zhang [9] studied this questions and obtained a series results, which answered the open questions. Recently, Zhang and L*ü* [10] considered the problem of f^n and $f^{(k)}$ sharing a small function and obtained the following results, which are the improvements and complements of above theorems.

Theorem D ([10]). *Let* $k(\geq 1)$, $n(\geq 1)$ *be integers and* f *be a non-constant meromorphic function. Also, let* $a = a(z)(\not\equiv 0, \infty)$ *be a small function such that* $T(r, a) = S(r, f)$, *as* $r \to \infty$. *Suppose that* f^n *and* $f^{(k)}$ *share a IM and*

$$4\overline{N}(r, f) + 2\overline{N}(r, \frac{1}{(f^n / a)'}) + \overline{N}(r, \frac{1}{f^{(k)}}) + 2N_2(r, \frac{1}{f^{(k)}}) < (\lambda + o(1))T(r, f^{(k)}),$$

or f^n *and* $f^{(k)}$ *share a CM and*

$$2\overline{N}(r, f) + \overline{N}(r, \frac{1}{(f^n / a)'}) + N_2(r, \frac{1}{f^{(k)}}) < (\lambda + o(1))T(r, f^{(k)}),$$

for $r \in I$, *where* I *is a set of infinite linear measure and* λ *satisfies* $0 < \lambda < 1$, *then* $\dfrac{f^{(k)} - a}{f^n - a} \equiv c$ *for some constant* $c \in C \setminus \{0\}$.

Theorem E ([10]). *Let* $k(\geq 1)$, $n(\geq 1)$ *be integers and* f *be a non-constant meromorphic function. Also, let* $a = a(z)(\not\equiv 0, \infty)$ *be a small function with respect to f. If* f^n *and* $f^{(k)}$ *share a IM and*

$$(2k + 6)\Theta(\infty, f) + 4\Theta(0, f) + 2\delta_{2+k}(0, f) > 2k + 12 - n,$$

or f^n *and* $f^{(k)}$ *share a CM and*

$$(k + 3)\Theta(\infty, f) + 2\Theta(0, f) + \delta_{2+k}(0, f) > k + 6 - n,$$

then $f^n = f^{(k)}$.

In the same paper, Zhang and L*ü* posed the following question: What will happen if f^n and $(f^{(k)})^m (m \geq 2)$ share a small function? In this paper, we consider this problem and obtain the following results, which answered the question.

Theorem 1. *Let* $k(\geq 1)$, $n(\geq 1)$, $m(\geq 2)$ *be integers and* f *be a non-constant meromorphic function. Also, let* $a = a(z)(\not\equiv 0, \infty)$ *be a small function such that* $T(r, a) = S(r, f)$, *as* $r \to \infty$. *Suppose that* f^n *and* $(f^{(k)})^m$ *share a IM and*

$$\frac{4}{m}\overline{N}(r, f) + \frac{2}{m}\overline{N}(r, \frac{1}{(f^n / a)'}) + \frac{5}{m}\overline{N}(r, \frac{1}{f^{(k)}}) < (\lambda + o(1))T(r, f^{(k)}), \quad (1)$$

or f^n *and* $(f^{(k)})^m$ *share a CM and*

$$\frac{2}{m}\overline{N}(r, f) + \frac{1}{m}\overline{N}(r, \frac{1}{(f^n / a)'}) + \frac{2}{m}\overline{N}(r, \frac{1}{f^{(k)}}) < (\lambda + o(1))T(r, f^{(k)}), \quad (2)$$

for $r \in I$, *where* I *is a set of infinite linear measure and* λ *satisfies* $0 < \lambda < 1$, *then* $\frac{(f^{(k)})^m - a}{f^n - a} \equiv c$ *for some constant* $c \in C \setminus \{0\}$.

Theorem 2. *Let* $k(\geq 1)$, $n(\geq 1)$, $m(\geq 2)$ *be integers and* f *be a non-constant meromorphic function. Also, let* $a = a(z)(\not\equiv 0, \infty)$ *be a small function with respect to* f. *If* f^n *and* $(f^{(k)})^m$ *share a IM and*

$$(3k + 6)\Theta(\infty, f) + 4\Theta(0, f) + 3\delta_{1+k}(0, f) > 3k + 13 - n, \quad (3)$$

or f^n *and* $(f^{(k)})^m$ *share a CM and*

$$(k + 3)\Theta(\infty, f) + 2\Theta(0, f) + \delta_{1+k}(0, f) > k + 6 - n, \quad (4)$$

then $f^n = (f^{(k)})^m$.

2. Some Lemmas

For the proof of our results, we need the following lemmas:

Lemma 1 ([2], [9]). *Let* $f(z)$ *be a non-constant meromorphic function, and* k *be a positive integer, then*

$$N_p(r, \frac{1}{f^{(k)}}) \leq N_{p+k}(r, \frac{1}{f}) + k\overline{N}(r, f) + S(r, f)$$

$$\leq (p + k)\overline{N}(r, \frac{1}{f}) + k\overline{N}(r, f) + S(r, f).$$

Lemma 2 ([1], [9]). *Let* $f(z)$ *be a non-constant meromorphic function, and* k *be a positive integer, then*

$$N(r, \frac{1}{f^{(k)}}) \leq N(r, \frac{1}{f}) + k\overline{N}(r, f) + S(r, f),$$

$$N(r, \frac{f^{(k)}}{f}) \leq \overline{N}(r, \frac{1}{f}) + k\overline{N}(r, f).$$

Lemma 3 ([4]). *Let $f(z)$ be a non-constant meromorphic function, and n be a positive integer. $P(f) = a_n f^n + a_{n-1} f^{n-1} + \cdots + a_1 f$, where a_i are meromorphic functions such that $T(r, a_i) = S(r, f)(i = 1, 2, \ldots, n)$, $a_n \not\equiv 0$. Then*

$$T(r, P(f)) = nT(r, f) + S(r, f).$$

3. Proof of the Theorems

Proof of the Theorem 1. Let $F(z) = \frac{f^n}{a}$, $G(z) = \frac{(f^{(k)})^m}{a}$, then $F - 1 = \frac{f^n - a}{a}$, $G - 1 = \frac{(f^{(k)})^m - a}{a}$. Since f^n and $(f^{(k)})^m$ share a IM(CM), F and G share 1 IM(CM) except the zeros and poles of $a(z)$. Define

$$H = (\frac{F''}{F'} - \frac{2F'}{F-1}) - (\frac{G''}{G'} - \frac{2G'}{G-1}). \tag{5}$$

We have the following two cases to investigate:

Case 1. $H \equiv 0$, integration yields

$$\frac{1}{F-1} = \frac{c}{G-1} + d, \tag{6}$$

where c and d are constants and $c \neq 0$. Next, we prove $d = 0$. If $\overline{N}(r, f) \neq S(r, f)$, then by (6), we get $d = 0$. And hence, in the following, we can suppose that $\overline{N}(r, f) = S(r, f)$. From (6), we know that F and G share 1 CM. We first assume that $d \neq 0$, then

$$\frac{1}{F-1} = \frac{d(G-1+c/d)}{G-1},$$

so

$$\overline{N}(r, \frac{1}{G-1+c/d}) = \overline{N}(r, F) = \overline{N}(r, f) + S(r, f) = S(r, f). \tag{7}$$

If $c \neq d$, by the second fundamental theorem and (7), we have

$$T(r, G) \leq \overline{N}(r, G) + \overline{N}(r, \frac{1}{G}) + \overline{N}(r, \frac{1}{G-1+c/d}) + S(r, G)$$

$$\leq \overline{N}(r, \frac{1}{G}) + S(r, G) \leq T(r, G) + S(r, G).$$

So

$$T(r, G) = \overline{N}(r, \frac{1}{G}) + S(r, G).$$

By Lemma 3, that is,

$$T(r, f^{(k)}) = \frac{1}{m}\overline{N}(r, \frac{1}{f^{(k)}}) + S(r, f).$$

This is contradiction with conditions (1) and (2). If $c = d$, then $\frac{1}{F-1} = \frac{cG}{G-1}$, then $(F-1-\frac{1}{c})G = -\frac{1}{c}$. By the definitions of F and G, we have

$$\frac{(f^{(k)})^m}{f^m} \equiv -\frac{a^2}{c}\frac{1}{f^m(f^n-(1+1/c)a)}.$$

So by Lemmas 1 and 3, we have

$$(m+n)T(r, f) = mT(r, \frac{f^{(k)}}{f}) + S(r, f)$$

$$\leq mN(r, \frac{1}{f}) + mk\overline{N}(r, f) + S(r, f)$$

$$\leq mT(r, f) + S(r, f).$$

This indicates $T(r, f) = S(r, f)$ since $n \geq 1$, which is a contradiction. Hence $d = 0$, so $\frac{G-1}{F-1} \equiv c$, i.e., $\frac{(f^{(k)})^m - a}{f^n - a} \equiv c$. This is just the conclusion of this theorem.

Case 2. $H \not\equiv 0$. From (5), it is to see that $m(r, H) = S(r, f)$.

Subcase 2.1. Suppose that f^n and $(f^{(k)})^m$ share a IM, in this case, F and G share 1 IM except the zeros and poles of $a(z)$. We have

$$\overline{N}(r, F) = \overline{N}(r, f) + S(r, f), \qquad \overline{N}(r, G) = \overline{N}(r, f) + S(r, f), \tag{8}$$

and by (5), we have

$$\begin{aligned} N(r, H) \leq\ & \overline{N}(r, F) + \overline{N}_{(2}(r, \frac{1}{F}) + \overline{N}_{(2}(r, \frac{1}{G}) + \overline{N}_L(r, \frac{1}{F-1}) \\ & + \overline{N}_L(r, \frac{1}{G-1}) + \overline{N}_0(r, \frac{1}{F'}) + \overline{N}_0(r, \frac{1}{G'}) + S(r, f), \end{aligned} \tag{9}$$

where $N_0(r, \frac{1}{F'})$ denotes the counting function of the zeros of F', which are not the zeros of F and $F-1$, and $\overline{N}_0(r, \frac{1}{F'})$ denotes its reduced form. In the same way, we can define $N_0(r, \frac{1}{G'})$ and $\overline{N}_0(r, \frac{1}{G'})$.

From the definitions of F and G, we get

$$\begin{aligned} N_{11}(r, \frac{1}{F-1}) &= N_{11}(r, \frac{1}{G-1}) + S(r, f),\ N_{(22}(r, \frac{1}{F-1}) \\ &= N_{(22}(r, \frac{1}{G-1}) + S(r, f). \end{aligned} \tag{10}$$

$$\begin{aligned} \overline{N}_L(r, \frac{1}{F-1}) &\leq N(r, \frac{1}{F-1}) - \overline{N}(r, \frac{1}{F-1}) \leq N(r, \frac{F}{F'}) \\ &\leq N(r, \frac{F'}{F}) + S(r, f) \leq \overline{N}(r, \frac{1}{F}) + \overline{N}(r, F) + S(r, f). \end{aligned} \tag{11}$$

$$\overline{N}(r, \frac{1}{F}) = \overline{N}(r, \frac{1}{G-1}) + S(r, f)$$

$$= N_{11}(r, \frac{1}{F-1}) + \overline{N}_{(22}(r, \frac{1}{F-1}) + \overline{N}_L(r, \frac{1}{F-1})$$

$$+ \overline{N}_L(r, \frac{1}{G-1}) + S(r, f). \tag{12}$$

Let z_0 be a common simple zero of $F-1$ and $G-1$, but $a(z_0) \neq 0, \infty$. By calculation, we know that z_0 is a zero of H, so

$$N_{11}(r, \frac{1}{F-1}) \leq N(r, \frac{1}{H}) + S(r, f) \leq T(r, H) + S(r, f)$$

$$\leq N(r, H) + S(r, f). \tag{13}$$

From (9), (10), (11), (12), and (13), we have

$$\overline{N}(r, \frac{1}{G-1}) \leq \overline{N}(r, F) + 2\overline{N}_L(r, \frac{1}{F-1}) + 2\overline{N}_L(r, \frac{1}{G-1})$$

$$+ \overline{N}_{(2}(r, \frac{1}{F}) + \overline{N}_{(2}(r, \frac{1}{G}) + \overline{N}_{(22}(r, \frac{1}{F-1})$$

$$+ \overline{N}_0(r, \frac{1}{F'}) + \overline{N}_0(r, \frac{1}{G'}) + S(r, f)$$

$$\leq \overline{N}(r, F) + 2\overline{N}(r, \frac{1}{F'}) + 2\overline{N}_L(r, \frac{1}{G-1})$$

$$+ \overline{N}_{(2}(r, \frac{1}{G}) + \overline{N}_0(r, \frac{1}{G'}) + S(r, f).$$

By the second fundamental theorem, we have

$$T(r, G) \leq \overline{N}(r, G) + \overline{N}(r, \frac{1}{G}) + \overline{N}(r, \frac{1}{G-1}) - \overline{N}_0(r, \frac{1}{G'}) + S(r, G)$$

$$\leq 2\overline{N}(r, G) + 2\overline{N}(r, \frac{1}{F'}) + \overline{N}(r, \frac{1}{G}) + \overline{N}_{(2}(r, \frac{1}{G})$$

$$+ 2\overline{N}_L(r, \frac{1}{G-1}) + S(r, f)$$

$$\leq 2\overline{N}(r, G) + 2\overline{N}(r, \frac{1}{F'}) + \overline{N}(r, \frac{1}{G}) + 2\overline{N}(r, \frac{1}{G'}) + S(r, f).$$

By Lemma 1, and noting that $\overline{N}(r, \frac{1}{G'}) = N_1(r, \frac{1}{G'})$, $\overline{N}(r, \frac{1}{(f^{(k)})^m})$ $= \overline{N}(r, \frac{1}{f^{(k)}})$, and $N_2(r, \frac{1}{(f^{(k)})^m}) = 2\overline{N}(r, \frac{1}{f^{(k)}})$ when $m \geq 2$, we get

$$T(r, G) \leq 4\overline{N}(r, f) + 2\overline{N}(r, \frac{1}{(f^n / a)'}) + 5\overline{N}(r, \frac{1}{f^{(k)}}) + S(r, f).$$

From above inequity and by Lemma 3, we obtain

$$T(r, f^{(k)}) \leq \frac{4}{m}\overline{N}(r, f) + \frac{2}{m}\overline{N}(r, \frac{1}{(f^n / a)'}) + \frac{5}{m}\overline{N}(r, \frac{1}{f^{(k)}}) + S(r, f),$$

which contradicts (1).

Subcase 2.2. Suppose that f^n and $(f^{(k)})^m$ share a CM, in this case, F and G share 1 CM except the zeros and poles of $a(z)$. From (5), we have

$$N(r, H) \leq \overline{N}(r, F) + \overline{N}_{(2}(r, \frac{1}{F}) + \overline{N}_{(2}(r, \frac{1}{G}) + \overline{N}_0(r, \frac{1}{F'})$$
$$+ \overline{N}_0(r, \frac{1}{G'}) + S(r, f). \tag{14}$$

Let z_0 be a common simple zero of $F - 1$ and $G - 1$, but $a(z_0) \neq 0, \infty$. By calculation, we know that z_0 is a zero of H, so

$$N_{1)}(r, \frac{1}{F-1}) \leq N(r, \frac{1}{H}) + S(r, f) \leq N(r, H) + S(r, f). \tag{15}$$

Noting that $N_{1)}(r, \frac{1}{F-1}) = N_{1)}(r, \frac{1}{G-1}) + S(r, f)$, so

$$\overline{N}(r, \frac{1}{G-1}) = N_{1)}(r, \frac{1}{F-1}) + N_{(2}(r, \frac{1}{F-1})$$
$$\leq \overline{N}(r, F) + N_{(2}(r, \frac{1}{F}) + N_{(2}(r, \frac{1}{G}) + N_{(2}(r, \frac{1}{F-1})$$
$$+ N_0(r, \frac{1}{F'}) + N_0(r, \frac{1}{G'}) + S(r, f).$$

By the second fundamental theorem, we can get

$$T(r, G) \leq \overline{N}(r, G) + \overline{N}(r, \frac{1}{G}) + \overline{N}(r, \frac{1}{G-1}) - \overline{N}_0(r, \frac{1}{G'}) + S(r, G)$$

$$\leq 2\overline{N}(r, G) + \overline{N}(r, \frac{1}{G}) + N_{(2}(r, \frac{1}{G}) + \overline{N}(r, \frac{1}{F'}) + S(r, f)$$

$$= 2\overline{N}(r, G) + N_{(2}(r, \frac{1}{G}) + \overline{N}(r, \frac{1}{F'}) + S(r, f).$$

Similarly, we have

$$T(r, f^{(k)}) \leq \frac{2}{m}\overline{N}(r, f) + \frac{1}{m}\overline{N}(r, \frac{1}{(f^n / a)'}) + \frac{2}{m}\overline{N}(r, \frac{1}{f^{(k)}}) + S(r, f),$$

this contradicts (2). Theorem 1 thus completely proved. □

Proof of the Theorem 2. The procedure is similar with the proof of Theorem 1, we define F and G as and (5) as above, and we also distinguish two cases to discuss:

Case 1. $H \not\equiv 0$.

Subcase 1.1. Suppose that f^n and $(f^{(k)})^m$ share a IM, then by the second fundamental theorem, we have

$$T(r, F) + T(r, G) \leq 2\overline{N}(r, F) + \overline{N}(r, \frac{1}{F}) + \overline{N}(r, \frac{1}{G}) + \overline{N}(r, \frac{1}{F-1})$$
$$+ \overline{N}(r, \frac{1}{G-1}) - \overline{N}_0(r, \frac{1}{F'}) - \overline{N}_0(r, \frac{1}{G'}) + S(r, f). \tag{16}$$

By the definitions of F and G, we have

$$\overline{N}(r, \frac{1}{F-1}) + \overline{N}(r, \frac{1}{G-1}) \leq \overline{N}_L(r, \frac{1}{F-1}) + N(r, \frac{1}{G})$$
$$+ N_{1)}(r, \frac{1}{F-1}) + S(r, f). \tag{17}$$

From (9), (13), (16), and (17), we deduce

$$T(r, F) \leq 3\overline{N}(r, F) + N_2(r, \frac{1}{F}) + N_2(r, \frac{1}{G})$$

$$+ 2\overline{N}_L(r, \frac{1}{F-1}) + \overline{N}_L(r, \frac{1}{G-1}) + S(r, f).$$

Applying (11), Lemmas 3 and 1 to the above inequality, we deduce

$$nT(r, f) \leq 6\overline{N}(r, f) + 4\overline{N}(r, \frac{1}{f}) + 3\overline{N}(r, \frac{1}{f^{(k)}}) + S(r, f)$$

$$\leq (6 + 3k)\overline{N}(r, f) + 4\overline{N}(r, \frac{1}{f}) + 3N_{k+1}(r, \frac{1}{f}) + S(r, f),$$

i.e.,

$$(6 + 3k)\Theta(\infty, f) + 4\Theta(0, f) + 3\delta_{k+1}(0, f) \leq 13 + 3k - n,$$

this contradicts (3).

Subcase 1.2. Suppose that f^n and $(f^{(k)})^m$ share a CM, then by the second fundamental theorem, we have

$$\overline{N}(r, \frac{1}{F-1}) + \overline{N}(r, \frac{1}{G-1}) = N_{1)}(r, \frac{1}{F-1}) + \overline{N}_{(2}(r, \frac{1}{F-1}) + \overline{N}(r, \frac{1}{G-1})$$

$$\leq N(r, H) + N_{(2}(r, \frac{1}{F-1}) + \overline{N}(r, \frac{1}{G-1})$$

$$\leq N(r, H) + N(r, \frac{1}{G-1}) + S(r, f)$$

$$\leq N(r, H) + T(r, G) + S(r, f). \tag{18}$$

From (14), (16), (18) above, and the second fundamental theorem, we have

$$T(r, F) \leq 3\overline{N}(r, F) + N_2(r, \frac{1}{F}) + N_2(r, \frac{1}{G}) + S(r, f).$$

Similarly, we have

$$nT(r, f) \leq 3\overline{N}(r, f) + 2\overline{N}(r, \frac{1}{f}) + 2\overline{N}(r, \frac{1}{f^{(k)}}) + S(r, f)$$

$$\leq (k+3)\overline{N}(r, f) + 2\overline{N}(r, \frac{1}{f}) + N_{k+1}(r, \frac{1}{f}) + S(r, f),$$

i.e.,

$$(k+3)\Theta(\infty, f) + 2\Theta(0, f) + \delta_{k+1}(0, f) \leq k + 6 - n,$$

this contradicts (4).

Case 2. $H \equiv 0$. We also have (6). Next, we distinguish two cases again:

Subcase 2.1. $d = 0$, that is, $F - 1 = \frac{1}{c}(G - 1)$. If $c \neq 1$, we have $G = c(F - 1) + 1$. By the second fundamental theorem, we have

$$T(r, F) \leq \overline{N}(r, F) + \overline{N}(r, \frac{1}{F}) + \overline{N}(r, \frac{1}{F + 1/c - 1}) + S(r, F)$$

$$\leq \overline{N}(r, F) + \overline{N}(r, \frac{1}{F}) + \overline{N}(r, \frac{1}{G}) + S(r, F),$$

i.e.,

$$nT(r, f) \leq \overline{N}(r, f) + \overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{f^{(k)}}) + S(r, f)$$

$$\leq \overline{N}(r, f) + \overline{N}(r, \frac{1}{f}) + N_{k+1}(r, \frac{1}{f}) + S(r, f),$$

i.e.,

$$\Theta(\infty, f) + \Theta(0, f) + \delta_{k+1}(0, f) \leq 3 - n.$$

Combining above inequity and the conditions of Theorem 2 yields

$$(3k+3)\Theta(\infty, f) + \Theta(0, f) > 4 + 3k + 3n,$$

or

$$(k+2)\Theta(\infty, f) + \Theta(0, f) > k + 3,$$

this is impossible with $\Theta(0, f) \leq 1$, $\Theta(\infty, f) \leq 1$. Hence $c = 1$ and $f^n = (f^{(k)})^m$.

Subcase 2.2. $d \neq 0$. By (6), we have $\overline{N}(r, f) = S(r, f)$, this imply $\Theta(\infty, f) = 1$. On the other hand, by (6), we have

$$\frac{c}{G-1} = -\frac{dF-(D+1)}{F-1},$$

so

$$\overline{N}(r, \frac{1}{F-(d+1)/d}) = \overline{N}(r, G) = S(r, f).$$

If $d \neq -1$, by the second fundamental theorem, we have

$$\begin{aligned} T(r, F) &\leq \overline{N}(r, F) + \overline{N}(r, \frac{1}{F}) + \overline{N}(r, \frac{1}{F-(1+d)/d}) + S(r, F) \\ &= \overline{N}(r, F) + S(r, f), \end{aligned}$$

i.e.,

$$nT(r, f) \leq \overline{N}(r, f) + S(r, f) \leq T(r, f) + S(r, f).$$

If $n \geq 2$, we obtain $T(r, f) = S(r, f)$. This is impossible. If $n = 1$, we have $\overline{N}(r, \frac{1}{f}) = T(r, f) + S(r, f)$, this implies $\Theta(0, f) = 0$. With the conditions of Theorem 2, we have $\delta_{k+1}(0, f) > 1$, it is impossible, and we get $d = -1$, then $\overline{N}(r, \frac{1}{F}) = S(r, f)$, i.e., $\overline{N}(r, \frac{1}{f}) = S(r, f)$, and from (6), we have $F(G-1-c) = -c$, this is, $(f^{(k)})^n((f^{(k)})^m - (1+c)\alpha) = -ca^2 \frac{(f^{(k)})^n}{f^n}$. So, we have

$$\begin{aligned} (m+n)T(r, f^{(k)}) &= nT(r, \frac{f^{(k)}}{f}) + S(r, f) = nN(r, \frac{f^{(k)}}{f}) + S(r, f) \\ &\leq n(k\overline{N}(r, f) + k\overline{N}(r, \frac{1}{f})) + S(r, f) = S(r, f). \end{aligned}$$

So $T(r, f^{(k)}) = S(r, f)$ and $T(r, \frac{f^{(k)}}{f}) = S(r, f)$. Hence

$$T(r, f) = T(r, \frac{f}{f^{(k)}}) + T(r, f^{(k)}) + S(r, f)$$

$$= T(r, \frac{f^{(k)}}{f}) + T(r, f^{(k)}) + S(r, f) = S(r, f).$$

This is impossible. Theorem 2 has been completely proved. □

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References

[1] R. Bruck, On entire functions which share one value with their first derivatives, Results Math. 30 (1996), 21-24.

[2] I. Lahiri, Uniqueness of a meromorphic function and its derivative, J. Inequal. Pure Appl. Math. 5(1) (2004), 9, Art. 20.

[3] L. P. Liu and Y. X. Gu, Uniqueness of meromorphic functions that share one small function with their derivatives, Kodai. Math. J. 27 (2004), 272-279.

[4] C. C. Yang, On deficiencies of differential polynomials II, Math. Z. 125 (1972), 107-112.

[5] C. C. Yang and H. X. Yi, Uniqueness Theory of Meromorphic Functions, Science Press, Beijing, 2006.

[6] L. Z. Yang, Solution of a differential equation and its applications, Kodai Math. J. 22 (1999), 458-464.

[7] K. W. Yu, On entire and meromorphic functions that share small functions with their derivatives, J. Inequal. Pure Appl. Math. 4(1) (2003), 7, Art. 21.

[8] Q. C. Zhang, The uniqueness of meromorphic functions with their derivatives, Kodai Math. J. 21 (1998), 179-184.

[9] Q. C. Zhang, Meromorphic function that shares one small function with its derivative, J. Inequal. Pure Appl. Math. 6(4) (2005), 13, Art. 116.

[10] Tongdui Zhang and Weiran L*ü*, Notes on a meromorphic function sharing one small function with its derivative, Complex Variables and Elliptic Equations 53(9) (2008), 857-867.

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